

STRING COSMOLOGY AND CHAOS

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We investigate Bianchi type IX "Mixmaster" universes within the framework of the low-energy tree-level effective action for string theory, which (when the "stringy" 2-form axion potential vanishes) is formally the same as the Brans-Dicke action with $\omega = -1$. We show that, unlike the case of general relativity in vacuum, there is **no Mixmaster chaos** in these string cosmologies. In the Einstein frame an infinite sequence of chaotic oscillations of the scale factors on approach to the initial singularity is impossible, as it was in general relativistic Mixmaster universes in the presence of stiff -fluid matter (or a massless scalar field). A finite sequence of oscillations of the scale factors approximated by Kasner metrics is possible, but it always ceases when all expansion rates become positive. In the string frame the evolution through Kasner epochs changes to a new form which reflects the duality symmetry of the theory. Again, we show that chaotic oscillations must end after a finite time. The need for **duality symmetry** appears to be incompatible with the presence of chaotic behaviour as $t \rightarrow 0$. We obtain our results using the Hamiltonian qualitative cosmological picture for Mixmaster models. We mention possible relations of this picture to diffeomorphism-independent methods of measuring chaos in general relativity. Finally, we discuss the effect of inhomogeneities and higher dimensions on the possible emergence of chaos within the string cosmology.

1 Chaos in Relativity and Cosmology

Chaotic systems arise in many areas of astronomy. Early studies of the stability of the solar system by Poincaré have led to the modern investigations of the motions of the satellites of Saturn and the chaotic evolution of planetary obliquities by Laskar¹. The modern resurgence of interest in chaotic dynamics was partly stimulated by the work of Hénon-Heiles on the orbits of stars in the

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disk of the Milky Way. These systems have been studied within the context of Newton's theory and all the classical criteria for defining chaos, in particular, non-vanishing Lyapunov exponents and metric entropy can be applied. However, general relativistic systems are different because they are automatically coordinate covariant and are best studied using covariant measures of chaos since Lyapunov exponents are explicitly time-dependent and also depend on the measure in the phase space which makes them also spatial diffeomorphism dependent.

1.1 Mixmaster Universe. Hamiltonian Approach

The paradigm for studies of general relativistic chaos in cosmology has been the anisotropic Bianchi type IX universe or Mixmaster universe². Different arguments for the appearance of chaos have been applied; the most transparent is the Hamiltonian approach³ which reduces the problem to the motion of a universe 'point' in a time-dependent, steep-walled minisuperspace potential. These methods were applied to show that a system describing the vacuum Bianchi IX model always allows the universe point to catch up with the expanding potential walls and make infinitely many bounces on the approach to the initial singularity. Then, since every time the universe point is scattered the uncertainty in the initial conditions grows, the system is chaotic. The Kasner indices fulfil the requirements

$$\sum_{i=1}^3 p_i = 1, \quad \sum_{i=1}^3 p_i^2 = 1. \quad (1)$$

These conditions, in particular, do not allow the universe to enter the regime in which it can be isotropic (there is no closed vacuum Friedmann universe). It was also found⁶ that the admission of stiff-fluid matter with equation of state $p = \rho$ changes the Kasner constraints to

$$\sum_{i=1}^3 p_i = 1, \quad \sum_{i=1}^3 p_i^2 = 1 - p_4^2, \quad (2)$$

with p_4 playing the role of the fourth Kasner index - a constant which come from the conservation law for the density. Such a choice admits a range of the indices to include isotropic Friedmann model ($p_i = 1/3$) and allows the monotonic evolution of the three scale factors ($-1/3 \leq p_1 \leq 1/3, 0 \leq p_2 \leq 2/3, 1/3 \leq p_3 \leq 1$). In the Hamiltonian picture this means that the universe point cannot be scattered against the potential walls infinitely many times and after making a finite number of chaotic oscillations the evolution becomes

monotonic and chaos ceases. Positive values of the indices correspond to the universe point having a normal component of its velocity towards the wall which is lower than the velocity of recession of the wall.

1.2 Establishing Chaos

Numerical studies of Bianchi IX universes confirmed its chaotic behaviour although Lyapunov exponents were shown to be asymptotically tending to zero through positive values in some cases⁷. In view of their non-covariant nature some other tools have been proposed. The first was to introduce the so-called ‘Lyapunov-like exponents’⁴ which can be expressed in terms of Ricci scalar and other invariants of the Riemann tensor. This characterisation relied on the reduction of Hamiltonian flows to geodesic flows on some Riemann manifolds. The second was to apply other invariant measures of chaos such as multifractal dimensions (information dimension, correlation dimension)⁵. Both methods can show in a coordinate independent way that the Mixmaster universe is chaotic.

2 String Cosmology and Chaotic Oscillations

2.1 String (Pre-Big-Bang) Cosmology

The low-energy effective-action for bosonic string theory was first presented in⁸ and the field equations, up to the first order in the inverse string tension α' in the *string frame*, are given by

$$\begin{aligned} R^\nu_\mu + \nabla_\mu \nabla^\nu \phi - \frac{1}{4} H_{\mu\alpha\beta} H^{\nu\alpha\beta} &= 0, \\ R - \nabla_\mu \phi \nabla^\mu \phi + 2 \nabla_\mu \nabla^\mu \phi - \frac{1}{12} H_{\mu\nu\beta} H^{\mu\nu\beta} &= 0, \\ \nabla_\mu (e^{-\phi} H^{\mu\nu\alpha}) &= 0, \end{aligned} \tag{3}$$

where ϕ is the dilaton field, $H_{\mu\nu\beta} = 6\partial_{[\mu} B_{\nu\beta]}$ is the field strength of the antisymmetric tensor $B_{\mu\nu} = -B_{\nu\mu}$ (usually called the axion). These equations with vanishing axion are the same as those of Brans-Dicke scalar-tensor gravity with a Brans-Dicke parameter $\omega = -1$. One recovers the Einstein limit from (3) when the axion vanishes and dilaton is constant. The equations (3) can be conformally transformed to the *Einstein frame*⁹ in which the field equations are those of Einstein relativity with two energy-momentum tensors for the dilaton (effectively stiff-fluid) and the axion.

Isotropic cosmology based on the equations (3) is totally different from the standard general relativistic cosmology since it admits a phase of expansion

for *negative* times as well as for positive times (with the singularity formally located at $t = 0$). Because of this, it was originally called *pre-big-bang* cosmology¹⁰. The novel feature is that it admits *superinflation* which is driven by the kinetic energy of the dilaton rather than by the potential energy of the inflaton, as it is in the ordinary inflationary picture. Superinflation (which is actually power-law inflation), however, appears for negative times after which the universe apparently approaches a stringy phase (strong curvature and coupling regime) where the effective equations (3) break down, before finally evolving towards a radiation-dominated expanding phase. Initially, the universe is approximated by the perturbative string Minkowski vacuum. This is different from standard inflationary picture in which the universe tunnels quantum mechanically from "nothing", then undergoes inflation, reheating and radiation-dominated expansion. The two types of the evolution of the scale factor (superinflation and radiation-dominated evolution) are *dual* to each other in the sense of string theory and this type of symmetry (including time-reflection) transforms large values of the scale factor at negative times into small values of the scale factor at positive times and vice versa. This feature is called scale factor duality¹⁰. Scale factor duality is an example of a more general continuous global $O(d,d)$ -symmetry of the string theory which is called T -duality⁹.

Anisotropic cosmologies based on the equations (3) have also been studied. This was done for universes of all Bianchi types and for Kantowski-Sachs. Because of the homogeneity there was no problem in admitting a homogeneous (i.e. spatially independent) dilaton field into all these geometries, although this was not the case with axion. By analogy with the electromagnetic field, one suspects that the admission of a spatially-independent antisymmetric tensor potential $B_{\mu\nu} = B_{\mu\nu}(t)$ should have elementary meaning (the so called 'elementary ansatz'). However, as it was proven by many authors¹¹, this ansatz necessarily requires a distinguished direction in space and prevents the universe from isotropization at late times in Bianchi type I and Kantowski-Sachs models. On the other hand, the elementary ansatz is not allowed in some Bianchi models. In particular, in Bianchi IX we have shown this is even the case for its axisymmetric subcase⁹. This led some authors to employ another ansatz in which the antisymmetric tensor field strength $H_{\mu\nu\rho}$ is time-dependent (the so called 'solitonic ansatz'). This is admitted by Bianchi IX geometries⁹. Lastly, for spatially homogeneous models with non-abelian symmetry one deals with the so-called non-abelian duality¹².

2.2 Is There Chaos?

The differences between Einstein theory and the effective bosonic string theory based on the equations (3) lead us to ask about the possible emergence of chaotic behaviour for anisotropic Bianchi IX type models in string theory. As mentioned in Section 1, the vacuum Bianchi IX model in general relativity is chaotic while the stiff-fluid (scalar field with no potential) model is not chaotic. In order to check whether there is chaos in stringy Bianchi IX models one should apply one of the methods discussed in Section 1.

It is easy to answer the question about chaos in the Einstein frame with homogeneous dilaton and axion under the solitonic ansatz since both axion and dilaton fields act as stiff fluids. One can show that the system oscillates only finite number of times and then chaos ceases⁹. However, as most of the physics of string cosmology should refer to the string frame, one needs to determine whether there are chaotic oscillations in that frame. Of course, one might argue that the physics should be frame-independent, but in the context of all changes of scale this is not necessarily a trivial statement. In order to confirm this we applied the Hamiltonian methods mentioned in Section 1.1. We calculated the velocity of a universe particle v_p moving against the potential walls and compared it with the maximum apparent velocity of a wall in the axion frame v_{max} (which is directly related to the string frame¹³)

$$v_p = \sqrt{\psi_{+\tau}^2 + \psi_{-\tau}^2} < v_{max} \approx \sqrt{\psi_{+\tau}^2 + \psi_{-\tau}^2 + \frac{1}{12}\phi_\tau^2 + \frac{1}{12}A^2e^{-2\phi}}, \quad (4)$$

with ψ_+ and ψ_- being the curvature anisotropies, subscript τ means the derivative with respect to a rescaled cosmic time $dt = d\tau e^{-\bar{\phi}}$ with $\bar{\phi}$ being the shifted dilaton¹³. Here, A is a constant determining the strength of the axion field (using the solitonic ansatz). The relation (4) shows that, unless both dilaton and axion vanish $\phi = A = 0$ (which is the case for the vacuum general relativity model) the universe particle cannot catch up with the walls infinitely many times. The angle of incidence of the universe particle moving towards a wall is too small sometimes and there are three regions in the corners of the potential which are excluded from scatterings. As a consequence, the universe makes a finite number of oscillations which stop at some moment and then the scale factors all evolve monotonically towards a curvature singularity⁹.

There are some other interesting points. Firstly, the Kasner conditions are different from those of vacuum or stiff-fluid case of Section 1, and read

$$\sum_{i=1}^3 p_i = 1 - p_4, \quad \sum_{i=1}^3 p_i^2 = 1, \quad (5)$$

where the fourth Kasner index p_4 arises from the field conservation law. This has consequences for the explicit duality-chaos relations which give some insight into the impossibility of the co-existence of both in one physical situation. There exist some chaotic Kasner-to-Kasner transitions that are also duality-related transitions of the form

$$p_1 \rightarrow -p_1, p_2 \rightarrow -p_2, p_3 \rightarrow -p_3, p_4 \rightarrow -(p_4 + 2), \quad (6)$$

which seem to prevent chaos. On the other hand, duality-related regions in the parameter space (4) are given by

$$-1 - \sqrt{3} \leq p_4 \leq -1, -\frac{1}{\sqrt{3}} \leq p_1 \leq \sqrt{\frac{2}{3}}, -\frac{2}{3} \leq p_2 \leq \frac{1}{2}\sqrt{\frac{2}{3}}, -1 \leq p_3 \leq -\frac{1}{2}\sqrt{\frac{2}{3}};$$

and

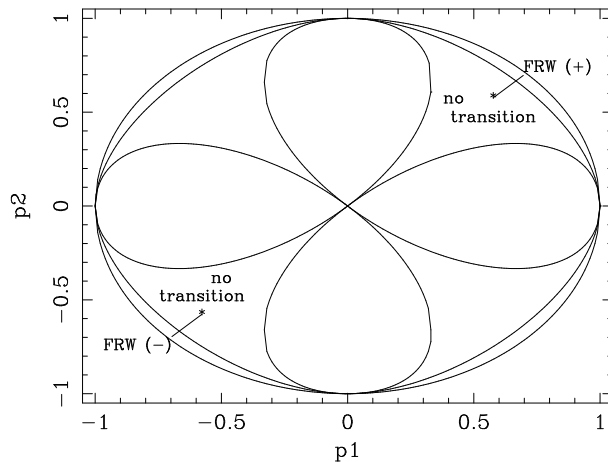
$$-1 \leq p_4 \leq -1 + \sqrt{3}, -\sqrt{\frac{2}{3}} \leq p_1 \leq \frac{1}{\sqrt{3}}, -\frac{1}{2}\sqrt{\frac{2}{3}} \leq p_2 \leq \frac{2}{3}, \frac{1}{2}\sqrt{\frac{2}{3}} \leq p_3 \leq 1.$$

Of course, bearing in mind the behaviour of the system in the Einstein frame, one can argue that this is just because we are dealing with the stiff-fluid. Secondly, as one can learn from the field equations with explicit Bianchi IX geometry⁹, that the dilaton field appears as a factor in front of the scale factors on the right-hand side of these equations, i.e.,

$$\begin{aligned} a^4 e^{-2\phi} &\propto t^{(2p_1 - p_4)} = t^{(1 + p_1 - p_2 - p_3)}, \\ b^4 e^{-2\phi} &\propto t^{(2p_2 - p_4)} = t^{(1 + p_2 - p_3 - p_1)}, \\ c^4 e^{-2\phi} &\propto t^{(2p_3 - p_4)} = t^{(1 + p_3 - p_1 - p_2)}, \end{aligned} \quad (7)$$

where a, b, c are the scale factors and t is the cosmic (comoving proper) time. This even restricts the possibilities for chaos to begin, as occurs for instance in five-dimensional vacuum Bianchi IX models¹⁵. The admissible values of the first two Kasner indices for chaotic oscillations to begin are given in Fig. 1. As one can easily see, the (dual) isotropic Friedmann cases ($p_4 = -1 + \sqrt{3}, p_1 = p_2 = p_3 = 1/\sqrt{3}$ and $p_4 = -1 - \sqrt{3}, p_1 = p_2 = p_3 = -1/\sqrt{3}$) are excluded.

Figure 1:



3 Perspectives

One of the simplest ways to recover chaos in low-energy string cosmology would be by the introduction of the dilaton potential (cosmological term) with possibly similar effect as the mass term in general relativistic scalar field cosmologies. This is also a situation which breaks duality.

Another way might be to appeal to higher dimensions or inhomogeneities. It is well-known that anisotropic 5-dimensional vacuum models are not chaotic¹⁵. This can be seen by analysing the generalized admissible ranges of Kasner indices

$$\sum_{i=1}^4 p_i = 1, \quad \sum_{i=1}^4 p_i^2 = 1 \quad (8)$$

similarly to our Fig.1 and by proving that the extra dimension slows the universe point which cannot catch up with the walls, finally leading to the monotonic (non-chaotic) expansion. If one enlarges the number of dimensions and also admits inhomogeneities or off-diagonal metric elements to the homogeneous diagonal Bianchi IX models, then the conditions (6) generalize to be

come

$$\sum_{i=1}^{D-1} p_i(x) = 1, \quad \sum_{i=1}^{D-1} p_i^2(x) = 1, \quad (9)$$

where D is the number of spacetime dimensions and x are spatial coordinates¹⁷. The result, in the Hamiltonian picture, is that the potential walls start rotating with the velocity of rotation going to zero on the approach to singularity and the universe point can actually be scattered infinitely many times, but this does not happen if the number of spacetime dimensions satisfies $D \geq 11$. The same effect of recovering chaos in multidimensional models appears in non-diagonal homogeneous Bianchi IX models of $D \leq 10$ ¹⁸.

These results show that the dimensional structure of type IX universes is non-trivial. The $D = 4$ case is the most complicated possible. Chaos occurs in diagonal vacuum models and is created by the intrinsically general relativistic parts of the Weyl curvature anisotropy which dominate the dynamics infinitely often on approach to a singularity, causing bounces. Physically, the motion of gravitational waves on a simple background space is curving up the space in the direction of propagation until the curvature back reaction reverses is motion. Note that for most of the time the 3-curvature of the type IX universe is actually negative. As the number of dimensions increases the situation becomes simpler. Off diagonal or inhomogeneous contributions to the metric are required to create chaos in the range $4 \leq D \leq 10$, but when the dimensionality exceeds this the evolution is no longer dominated by the intrinsically general relativistic Weyl curvature effects: the Newtonian parts of the gravitational field dominate on approach to the singularity. We should also note that one must distinguish between Mixmaster models with full $SO(D-1)$ invariance and those with a product manifold structure which behave effectively as if they have fewer dimensions. These features suggest further investigations since superstring theories are formulated in $D = 10$ spacetime dimensions, while M -theory with its low-energy supergravity limit is formulated in $D = 11$ dimensions. Unfortunately, so far there appears to be no specific link between the causes of the disappearance of chaos in $D > 10$ general relativity and the finiteness conditions that pick out $D = 10$ string theories. The disappearance of chaos in $D = 11$ dimensions might suggest that M -theory has a simpler structure in this particular respect. However, this might not necessarily be the case since its subsystems include Yang-Mills fields which are chaotic because of the colour charges even in axisymmetric Bianchi type I geometries¹⁹ but this chaos (unlike the Mixmaster phenomenon) is not general relativistic in origin. This suggests that we include other string modes together with a range of compactification schemes¹⁶ in order to answer the question about the

existence of chaos in string cosmology and its generalizations.

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Appendix

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